Exam Boundary Layers

June 22, 2016: 9.00-12.00.

This exam has 5 problems. Each problem is worth 2 points. Note: The final grade is increased by 1 point if the homework assigments have been done properly. Write on each page your name and student number. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by work. Success.

1. (a) (1 pt) Derive the boundary-layer (BL) equations from the steady, two-dimensional, incompressible Navier-Stokes equations.
   (b) (1/4 pt) Specify the boundary conditions for the BL-equations
   (c) (3/4 pt) To find 'analytic' solutions of the BL-equations simularity solutions are sought, inspired by the method of separations of variables. Here we look for solutions of the form
   $$u(x, y) = u_e(x) f(\eta)$$
   where $\eta = y/L(x)$. Show that this approach can be succesfull if
   $$u_e \sim U\, x^p \quad L \sim \sqrt{(\nu/U)}\, x^q \quad \text{with} \quad q = \frac{1-p}{2}$$

2. (2 pt) Find a composite expansion of
   $$\epsilon y'' + y' - y = 0,$$
   for $0 < x < 1$, where the boundary conditions are
   $$y(0) = 0 \quad \text{and} \quad y(1) = -1$$
   NB ϵ is a small positive number.

3. (a) (3/4 pt) Show that the total shear stress in a turbulent channel flow is given by
   $$\tau(y) = \tau_w \left(1 - \frac{y}{\delta}\right),$$
   where τ_w denotes the wall shear stress and the channel walls are at $y = 0$ and $y = 2\delta$, respectively.
   (b) (3/4 pt) In the laminar sublayer the shear stress is dominated by the molecular contribution $\tau = \mu \partial_y u$. As this layer is very thin the shear stress does not differ much from its value at the wall, τ_w. Show that this leads to
   $$u^+ = y^+$$
   (and give the definitions of u^+ and y^+).

(c) (1/2 pt) The flow in the inner layer is dominated by turbulent mixing. The mixing length ℓ is taken proportional to the disitance to the wall: $\ell = \kappa y$, where κ is a dimensionless constant. Show that

$$u^+ = \frac{1}{\kappa}\ln(y^+) + C$$

where C is constant.

4. The boundary layer equations have a parabolic character where the x-direction is time-like.

   (a) (1/4 pt) Explain this in case $v = 0$.

   The heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

   is a prototype of an parabolic PDE.

   (b) (3/4 pt) Write the heat equation as a system of first-order PDEs and discretize this system on a uniform grid using the box method of Keller. Show that this yields

$$\begin{aligned} &\frac{1}{4}\left(u_{i+1}^{n+1} + 2u_i^{n+1} + u_{i-1}^{n+1}\right) - \frac{1}{4}\left(u_{i+1}^{n} + 2u_i^{n} + u_{i-1}^{n}\right) \\ = &\frac{d}{4}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \frac{d}{4}\left(u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right) \end{aligned}$$

   (c) (1/2 pt) Determine the amplification factor of the Keller-box scheme in (b).

   (d) (1/2 pt) Show that the amplification factor goes to -1 if $d \to \infty$, and discuss this.

5. In large-eddy simulation (LES), the larger unsteady turbulent motions are directly represented, whereas the effects of the small-scale motions are modelled. To that end, the full velocity field u is decomposed into $u = \overline{u} + u'$, where $\overline{u}$ denotes a spatially filtered velocity field.

   (a) (1/4 pt) Give an example of a filter that is used in LES.

   (b) (1 pt) Derive the equations that govern the motion of the larger eddies from the incompressible Navier-Stokes equations, i.e., derive the right-hand side of

$$\frac{\partial \overline{u}}{\partial t} = \cdots$$

   (c) (3/4 pt) Formulate the LES closure problem.